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Caroline Lawless, Julyan Arbel

## ► To cite this version:

Caroline Lawless, Julyan Arbel. Chinese restaurant process from stick-breaking for Pitman-Yor. Bayesian learning theory for complex data modelling Workshop, Sep 2018, Grenoble, France. pp.1. hal-01950662

**HAL Id: hal-01950662**

**<https://hal.science/hal-01950662>**

Submitted on 11 Dec 2018

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# Chinese restaurant process from stick-breaking for Pitman–Yor

CAROLINE LAWLESS AND JULYAN ARBEL



UNIV. GRENoble ALPES, INRIA, CNRS, LJK, 38000 GRENoble,

FRANCE

## INTRODUCTION

- The Chinese restaurant process and the stick-breaking process are the two most commonly used representations of the Pitman–Yor process.
- However, the usual proof of the connection between them is indirect.
- [Miller \(2018\)](#) proved directly that the stick-breaking process gives rise to the Chinese restaurant process representation of the Dirichlet process.
- The Dirichlet process is a special case of the Pitman–Yor process.
- We extend Miller’s proof to Pitman–Yor process random measures.

## PITMAN–YOR & DIRICHLET PROCESSES

- The Dirichlet Process (DP) and the Pitman–Yor process (PY, [Pitman and Yor, 1997](#)) are discrete random probability measures.
- The PY is parametrized by  $d \in (0, 1)$ ,  $\alpha > -d$ , and a base probability measure  $P_0$ . The DP is recovered by letting  $d = 0$ .
- The stick-breaking representation ([Sethuraman, 1994](#)) is given by

$$v_i \sim \begin{cases} \text{Beta}(1, \alpha) & \text{for DP} \\ \text{Beta}(1 + d, \alpha + id) & \text{for PY} \end{cases}$$

$$\pi_k = v_k \prod_{i=1}^{k-1} (1 - v_i), \phi_k \stackrel{\text{iid}}{\sim} P_0.$$

We define the random process  $P$  by

$$P = \sum_{i=1}^{\infty} \pi_i \delta_{\phi_i}.$$

- The Chinese restaurant process ([Antoniak, 1974](#)) is the distribution induced on random partitions  $\mathbf{C}$  given by

$$P(\mathbf{C} = C) = \begin{cases} \frac{\alpha^{|\mathbf{C}|} \Gamma(\alpha)}{\Gamma(n + \alpha)} \prod_{c \in C} \Gamma(|c|) & \text{for DP} \\ \frac{d^t (\frac{\alpha}{d})_{(t)}}{(\alpha)_{(n)}} \prod_{j=1}^t (1 - d)_{(|c_j| - 1)} & \text{for PY.} \end{cases}$$

## THEOREM

Suppose  $\pi$  follows the [PY stick-breaking](#), and

$$\mathbf{z}_1, \dots, \mathbf{z}_n | \pi = \pi \stackrel{\text{iid}}{\sim} \pi, \text{ that is, } \mathbb{P}(\mathbf{z}_i = k | \pi) = \pi_k,$$

and  $\mathbf{C}$  is the partition of  $[n]$  induced by  $\mathbf{z}_1, \dots, \mathbf{z}_n$ . Then  $\mathbf{C}$  follows the [PY Chineses restaurant process](#).

## TECHNICAL LEMMAS

Our proof relies on the following lemmas, which here we will state without proof. Let us abbreviate  $\mathbf{z} = (z_1, \dots, z_n)$ . Given  $\mathbf{z} \in \mathbb{N}^n$ , let  $C_{\mathbf{z}}$  denote the partition  $[n]$  induced by  $\mathbf{z}$ . We define  $m(\mathbf{z}) = \max \{z_1, \dots, z_n\}$ , and  $g_k(\mathbf{z}) = \#\{i: z_i \geq k\}$ .

**Lemma 1** For any  $\mathbf{z} \in \mathbb{N}^n$ ,

$$\mathbb{P}(\mathbf{z} = \mathbf{z}) = \frac{1}{(\alpha)_{(n)}} \prod_{c \in C_{\mathbf{z}}} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \prod_{k=1}^{m(\mathbf{z})} \frac{\alpha + (k - 1)d}{g_k(\mathbf{z}) + \alpha + (k - 1)d}.$$

**Lemma 2** For any partition  $C$  of  $[n]$ ,

$$\sum_{\mathbf{z} \in \mathbb{N}^n} \mathbb{1}(C_{\mathbf{z}} = C) \prod_{k=1}^{m(\mathbf{z})} \frac{\alpha + (k - 1)d}{g_k(\mathbf{z}) + \alpha + (k - 1)d} = \frac{d^t (\frac{\alpha}{d})_{(t)}}{\prod_{c \in C} (|c| - d)}.$$

## PROOF OF THEOREM

$$\begin{aligned} \mathbb{P}(\mathbf{C} = C) &= \sum_{\mathbf{z} \in \mathbb{N}^n} \mathbb{P}(\mathbf{C} = C | \mathbf{z} = \mathbf{z}) \mathbb{P}(\mathbf{z} = \mathbf{z}) \\ &\stackrel{(a)}{=} \sum_{\mathbf{z} \in \mathbb{N}^n} \mathbb{1}(C_{\mathbf{z}} = C) \frac{1}{(\alpha)_{(n)}} \prod_{c \in C_{\mathbf{z}}} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \prod_{k=1}^{m(\mathbf{z})} \frac{\alpha + (k - 1)d}{g_k(\mathbf{z}) + \alpha + (k - 1)d} \\ &= \frac{1}{(\alpha)_{(n)}} \prod_{c \in C_{\mathbf{z}}} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \sum_{\mathbf{z} \in \mathbb{N}^n} \mathbb{1}(C_{\mathbf{z}} = C) \prod_{k=1}^{m(\mathbf{z})} \frac{\alpha + (k - 1)d}{g_k(\mathbf{z}) + \alpha + (k - 1)d} \\ &\stackrel{(b)}{=} \frac{1}{(\alpha)_{(n)}} \prod_{c \in C_{\mathbf{z}}} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \frac{d^t (\frac{\alpha}{d})_{(t)}}{\prod_{c \in C} (|c| - d)} \\ &\stackrel{(c)}{=} \frac{1}{(\alpha)_{(n)}} \prod_{c \in C} (1 - d)_{(|c| - 1)} \prod_{c \in C} (|c| - d) \frac{d^t (\frac{\alpha}{d})_{(t)}}{\prod_{c \in C} (|c| - d)} \\ &= \frac{d^t (\frac{\alpha}{d})_{(t)}}{(\alpha)_{(n)}} \prod_{j=1}^t (1 - d)_{(|c_j| - 1)} \end{aligned}$$

where (a) is by Lemma 1, (b) is by Lemma 2, and (c) is since  $\Gamma(|c| + 1 - d) = (|c| - d)\Gamma(|c| - d)$ .

## FURTHER RESEARCH

- The Dirichlet process and the Pitman–Yor process are only special cases of a broad class of random measures called Gibbs-type random measures.
- An interesting further study would be to investigate the possibility of extending this proof to Gibbs-type random measures.

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